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Equivalence of Deviau's, Hestenes', and Parra's Formulations of Dirac Theory[1]

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Daviau showed the equivalence of matrix Dirac theory, formulated within a spinor bundle $\mathcal{S}_x \simeq \mathbb{C}_x^4$, to a Clifford algebraic formulation within space Clifford algebra $\mathcal{C}\ell(\mathbb{R}^3, \delta) \simeq \mathbf{M}_2(\mathbb{C}) \simeq \mathcal{P} \simeq$ Pauli algebra (matrices) $\simeq \mathbb{H} \oplus \mathbb{H} \simeq$ biquaternions. We will show, that Daviau's map θ: $\mathbb{C}^4 \mapsto \mathrm{M}_2(\mathbb{C})$ is an isomorphism. It is shown that Hestenes' and Parra's formulations are equivalent to Daviau's Clifford algebra formulation, which uses outer automorphisms. The connection between different formulations is quite remarkable, since it connects the left and right action on the Pauli algebra itself viewed as a bi-module with the left (resp. right) action of the enveloping algebra $\mathcal{P}^e \simeq \mathcal{P} \otimes \mathcal{P}^T$ on $\mathcal{P}$. The isomorphism established in this article and given by Daviau's map does clearly show that right and left actions are of similar type. This should be compared with attempts of Hestenes, Daviau, and others to interprete the right action as the iso-spin freedom.

1. INTRODUCTION

A few months after the publication of Dirac's first paper (Dirac, 1928) Charles Galton Darwin tried to re-express the *strange new objects* called *half vectors* by Pauli (Pauli, 1933) and spinors due to Paul Ehrenfest—according to B.L. van der Waerden (see Budinich *et al.*, 1988)—with help of tensors (Darwin, 1928). He did not fully succeed in obtaining an equivalence by writing down complex tensor equations which yield Dirac's theory "twice over"—with a doubling of degrees of freedom from complexification—see Parra (1996) for a detailed review on this topic. Madelung, trying the same transcription, essentially reproduced Darwin's results, most likely without knowing them (Madelung, 1929). Also Fock and Ivanenko (Fock, 1929; Fock

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et al., 1929) did very important work on the geometric relations behind the γ-algebra introduced by Dirac. De Broglie and his school developed a very valuable and complete picture of the Dirac fluid—a tensor description of the Dirac field—and its hydrodynamics (Yvon, 1940; Takabayasi, 1957). This reasoning has a revival in recent times because of the improved tool of Clifford algebra now available (Rylov, 1995).

The historical development abandoned the attempt to find a geometric—and thereby tensorial—description of the Dirac field. There seemed to be a tendency to concrete calculations which on the one hand were extremely successful and on the other hand could be performed without an elaborated interpretation by applying simply the rules of γ-algebra, see discussion in (Isham, 1995). A quantum theory *had (has?) to be* interpreted within a statistical picture. It was simply out of the imagination of that time to search for such an explanation or even to connect geometry with spinor variables.

Neither the physicists Pauli and Dirac nor the mathematicians Weyl, Jordan, von Neumann, and others cited substantially seemed to have known the work of Grassmann, Clifford, Klein, Cayley, Hamilton, and other algebraists of the 19th century. If some of their formulas and results were acknowledged—the quaternions, *e.g.*, were well known to be isomorphic to Pauli matrices—this was done in a technical sense. The geometric origin of hypercomplex number systems was unknown or ignored and thus lost for further development of the theory. One result of this missed opportunity—in the sense of Dyson (1972)—was the thereby obtained "interpretation" of spinors, which became artificial objects in an *abstract spin space* or an *inner spin space* and had, thusly, no physical counterpart in the "real world."

The situation changes with the appearance of the writings of David Hestenes (Hestenes, 1966), see references in (Hestenes, 1997). He recovered again the geometric origin of spinor objects and the formerly well known connection of (metric) space and certain algebras. Hestenes gave a geometrical motivated treatment of real Dirac theory in his book *Space Time Algebra* (Hestenes, 1966). The reformulation of Dirac's theory in real (!) space time algebra $\mathcal{C}\ell(\mathbb{R}^4, \eta)$, $\eta = \mathrm{diag}(1, -1, -1, -1)$ is the starting point of a host of new insights into the interplay between geometry, algebra, and physics. Hestenes' reformulation was also the starting point of Daviau's consideration which lead to a algebraic ($\simeq \mathcal{C}\ell(\mathbb{R}^3, \delta)$), $\delta = \mathrm{diag}(1, 1, 1)$ formulation of Dirac theory.

A discussion on the proper interpretation of spinorial objects in either geometrical or statistical settings, lead to a large number of slightly different notations of spinors; *e.g.*, spinor modules $S_x \simeq \mathbb{C}_x^{2n}$, operator or Hestenes spinors $\simeq \mathcal{C}\ell^+_{p,q}$, ideal spinors $\simeq \mathcal{C}\ell f$, f an primitive idempotent element, algebraic spinors and the spin Clifford bundle–isomorphism classes of ideal spinors to geometrically equivalent idempotents' –*etc.* If Clifford algebra provides us the *universal language for mathematics and physics* (Hestenes, 1986), we have to give exact and unambiguous notations of physical objects and of their exact mathematical design.

Hestenes in succeeding to write down a *real* Dirac theory within $\mathcal{C}\ell_{1,3}$ translated the non-geometrical $i = \sqrt{-1}$ into the *right action* of $\gamma_2\gamma_1$, recall $(\gamma_2\gamma_1)^2 = -\gamma_1^2\gamma_2^2 = -1$. But right actions mix different left ideals related to different idempotents, while left action remains in the same left ideal. Rodrigues *et al.* introduced, therefore, the spin Clifford bundle and algebraic spinors, in which spinors or even better algebraic spinors are defined as equivalence classes of ideals which belong to geometrically equivalent idempotents (Rodrigues *et al.*, 1996, *cf.* also De Leo *et al.*, 1999). Such idempotents are conjugated to one another within the Clifford–Lipschitz group Γ by $e' = ueu^{\sim}$, $u \in \Gamma$, $^{\sim}$ the reversion map, and are, therefore, members of the same group orbit. To obtain a mathematical clear picture one should then translate the Dirac–Hestenes spinors into the quotient space $DH \simeq \mathcal{C}\ell_{1,3}/\Gamma$ (as linear space) to be not troubled with the probably ill chosen representants. This consideration should, however, be compared with the approach of Parra to Dirac–Hestenes spinors and his illuminating explanation of the equivalence classes and their relations to the Wigner definition of a *particle* as an irreducible representation of the Poincaré group (Parra, 1996).

In this paper, we study the map from Dirac matrix theory onto Clifford algebra used by Daviau. It is shown that a special option of Parra's formulation corresponds to Hestenes'. The equivalence of Hestenes', *sic.* Parra's, formulation to the Clifford algebraic formulation of Daviau is demonstrated. The correct identification to Parra's options is given.

Our aim is *not* to deal with the issue of representations. Our equivalence proof connects different abstract Clifford algebras, pointing out joices in the set-up of abstract algebra necessary due to identify sub-algebras, *etc.*, correctly. A morphology of spinor types can be found in Figueiredo *et al.* (1990).

We cannot appreciate every work concerned with Clifford algebraic formulations of Dirac theory for lack of space, one important paper may be added here (Baylis, 1997).

Our analysis unmasks a close connection between the ordinary spinor module $S_x \simeq \mathbb{C}_x^4$ which is equivalent to a formulation by ideal spinors in $\mathcal{C}\ell_{4,1}$, since $\mathcal{C}\ell_{4,1} \cong \mathbf{M}_4(\mathbb{C})$ which is actually used by physicists. Daviau's map furthermore shows up a correspondence of left actions on $\mathbb{C}_x^4$ spinors to homomorphisms of $\mathcal{P}$, which can be written as uxv, $u, v, x, \in \mathcal{P}$. If one defines the enveloping algebra $\mathcal{P}^e$ as in Hahn (Hahn, 1994), $\mathcal{P}^e \cong \mathcal{P} \otimes \mathcal{P}^T$, where $\mathcal{P}^T$ denotes the right module or transposed module, it is easily seen, that the $\mathcal{P}^e$ left action is equivalent to the $\mathcal{P}$-bi-module structure by writing $\mathcal{P}^e \bullet \mathcal{P} \mapsto \mathcal{P}$, $x \otimes y^T \bullet z = xzy$. We have therefore to consider left and right actions on $\mathcal{P}$, as Daviau did. This makes $\mathcal{P}$ a $\mathcal{P}$-bi-module. This bi-module structure is crucial for further investigations of the enveloping algebra $\mathcal{P}^e$ of Clifford algebras, which will be given elsewhere, and for a thoughtful interpretation of left and right actions in Clifford algebras. There is a wide-spread thinking about the meaning of right actions, (see Hestenes, 1967; Daviau, 1998b; Fauser *et al*, 2000c).

2. THE DAVIAU MAP θ: $\mathbb{C}^4 \mapsto \mathcal{C}\ell_{3,0}$

2.1 Definition of the Daviau Map

Daviau changed his notation and got rid of his cyclic permuted σ-matrices in a new work (Daviau, 1998a), however, we stay with his old notations. We start with the Dirac equation in its matrix representation due to Bjorken and Drell (Bjorken *et.al.*, 1964)

$$-i\gamma^\mu\partial_\mu \Psi + qA^\mu\gamma_\mu \Psi + m\Psi = 0. \tag{1}$$

We have m, q real constants, $i = \sqrt{-1}$ the usual complex unit, $\partial_\mu := \partial/\partial x^\mu$ the partial derivatives with respect to a local holonom coordinate system, A^μ real components of an external vector potential, Ψ is the Dirac spinor of $\mathbb{C}^4$ valued functions of the (tangent) Minkowski space and finally γ_μ the Dirac matrices in Dirac representation

$$\gamma_0 = \gamma^0 := \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \gamma_k = -\gamma^k := \begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix} \mathbb{1} := \mathbb{1}_{2\times 2}$$

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

It is an easy task to translate the Dirac equation into a set of eight real coupled differential equations (Parra, 1992),

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} := \begin{pmatrix} a + ie \\ -g - if \\ d + ih \\ b + ic \end{pmatrix}$$

with $a, \ldots, h$: $(M, \eta) \mapsto \mathbb{R}$ real valued functions. Here one does no longer insist on the "spinorial" character of the object in favor of playing with components and forgetting about transformation properties (Parra, 1996; Parra, 1992).

One has to consider the Pauli algebra or Clifford algebra $\mathcal{C}\ell_{3,0} \simeq \mathcal{P}$. This algebra is isomorphic to the full matrix algebra $M_2(\mathbb{C})$ and thus eight dimensional over the reals, $\dim_{\mathbb{R}} \Psi = 8 = \dim \mathcal{P} = \dim_{\mathbb{R}} M_2(\mathbb{C})$.

The aim of the Daviau map is to give an isomorphism from $\mathbb{C}_x^4 \rightarrow$ coordinates $\rightarrow \mathbf{M}_2(\mathbb{C})$ which is also a morphism of the *algebraic* structure. One could call such a map a Dirac-morphism.

By letting

$$u := a + ih,\ v := f + ib,\ w := c + ig,\ t := d + ie$$

$$\phi_1 := u + w,\ \phi_2 := t + v,\ \phi_3 := t - v,\ \phi_4 := u - w$$

$$\phi_D = \begin{pmatrix} \phi_1 & \phi_3 \\ \phi_2 & \phi_4 \end{pmatrix} \in M_2(\mathbb{C}) \simeq \mathcal{P} \simeq \mathcal{C}\ell_{3,0}, \tag{4}$$

we obtain a map $\theta\colon \mathbb{C}^4 \mapsto M_2(\mathbb{C})$. Introducing then (note our indexing)

$$\nabla := \partial_0 + \vec{\partial},\ \vec{\partial} := \sigma_2\partial_1 + \sigma_3\partial_2 + \sigma_1\partial_3$$

$$A := A_0 + \vec{A},\ \vec{A} := A^1\sigma_2 + A^2\sigma_3 + A^3\sigma_1$$

$$\phi^* := \begin{pmatrix} \overline{\phi}_4 & -\overline{\phi}_2 \\ -\overline{\phi}_3 & \overline{\phi}_1 \end{pmatrix} = \sigma_2\overline{\phi}\sigma_2$$

$$i := \sigma_1\sigma_2\sigma_3,\ [i, X] \quad = 0\ \forall X \in \mathcal{P}, \tag{5}$$

we obtain the space Clifford or Pauli algebraic form of Dirac's equation due to Daviau:

$$\nabla\phi i\sigma_1 = m\phi^* + qA\phi. \tag{6}$$

Daviau showed, that all transformation properties and requirements are fulfilled within this picture, making his map finally a Dirac-morphism preserving the algebraic structure of Dirac theory. A Lagrangian formulation is also possible. Using the above given representation of Pauli matrices (2) one can reconstruct an algebraic expression of the $M_2(\mathbb{C})$ matrix ϕ_D. From (4) we find a form of the Daviau spinor,

$$\begin{aligned} \phi_D &= \begin{pmatrix} u + w & t - v \\ t + v & u - w \end{pmatrix} \\ &= \begin{pmatrix} a + c + i(h + g) & d - f + i(e - b) \\ d + f + i(e + b) & a - c + i(h - g) \end{pmatrix} \\ &= a\mathbb{1} + d\sigma_1 + b\sigma_2 + c\sigma_3 + ei\sigma_1 - fi\sigma_2 + gi\sigma_3 + hi. \end{aligned} \tag{7}$$

2.2 Hestenes Equation

We may further notice, that since $\dim \mathcal{C}\ell_{1,3} = 16$ and $\dim \mathcal{C}\ell_{1,3}^+ = 8$, $\mathcal{C}\ell_{1,3}^+$ may also be used as a target for a map $H\colon \mathbb{C}^4 \mapsto \mathcal{C}\ell_{1,3}^+$. This algebra

$\mathcal{C}\ell^+$, called even subalgebra, consist of Dirac–Hestenes operator spinors and has in a natural manner a bimodule structure under the action of even elements. With the above choice of names for the real spinor components (3) we obtain the correspondence using $\gamma_{ij} := \gamma_i\gamma_i$, $\Sigma_i := \gamma_i\gamma_0$, $i := \Sigma_1\Sigma_2\Sigma_3 = \gamma_{0123}$:

$$\begin{aligned}\Psi_H &= a + b\gamma_{10} + c\gamma_{20} + d\gamma_{30} + e\gamma_{21} + f\gamma_{23} + g\gamma_{13} + h\gamma_{0123}. \\ &= a + b\sum\nolimits_1 + c\sum\nolimits_2 + d\sum\nolimits_3 - fi\sum\nolimits_1 + gi\sum\nolimits_2 + ei\sum\nolimits_3 + hi \end{aligned} \tag{8}$$

Where we have used the identities

$$i\sum\nolimits_1 = i\gamma_{10} = -\gamma_{23},\; i\sum\nolimits_2 = i\gamma_{20} = \gamma_{13},\; i\sum\nolimits_3 = i\gamma_{30} = \gamma_{21} \tag{9}$$

and anticipated the names of the variables in an appropriate manner to fit into the Daviau scheme. The translated Dirac equation reads ($m = m_0c/\hbar$, $q = e/\hbar c$, $\partial = \gamma^\mu\partial_\mu$, $A = \gamma^\mu A_\mu$)

$$\partial\Psi_H\gamma_{21} = m\Psi_H\gamma_0 + qA\Psi_H, \tag{10}$$

which is the famous Dirac–Hestenes equation and representation free. The elements on the right hand side of Ψ_H describe the spin bivector $S := \gamma_{21}$ and the "particles" (local) velocity $v := \gamma_0$—a time-like vector measuring proper-time—and do *not* fix a representation. For a discussion of the relation between quantum logic, measurement, and the choice of a time-like direction in Dirac theory see Haft (1996) and Saller (1996).

We may left multiply (10) by $-\gamma_0$ which turns the equation (beside the mass term) into the space part of the algebra. Using (9) and

$$\begin{aligned}-\gamma_0\partial &= -\gamma^0\gamma^\mu\partial_\mu = \Sigma_\mu\,\partial_\mu \\ -\gamma_0 A &= -\gamma^0\gamma^\mu A_\mu = \Sigma_\mu\,A_\mu\end{aligned} \tag{11}$$

we remain with

$$\begin{aligned}\sum\nolimits_\mu\partial_\mu\Psi_H i\sum\nolimits_3 &= -m\gamma_0\Psi_H\gamma_0 + q\sum\nolimits_\mu A_\mu\Psi_H \\ \sum\nolimits_\mu\partial_\mu\Psi_H i\sum\nolimits_3 &= -m\Psi_H^\dagger + q\sum\nolimits_\mu A_\mu\Psi_H,\end{aligned} \tag{12}$$

which is written now within the space sector only. The transformation $\Psi_H^\dagger = \gamma_0\Psi_H\gamma_0$ represents the hermitian adjoint, which is not an inner automorphism of the Pauli algebra isomorphic to $\mathcal{C}\ell^+_{1,3}$, as indicated by the odd element γ_0.

This form of the Dirac–Hestenes' formulation will be needed in the proof of the isomorphy to Daviau's formulation below.

2.3. Parra's Analysis of Dirac Theory

Parra analyzed the Dirac equation also in terms of a real set of eight differential equations (Parra, 1992). Like Darwin and Madelung he afterwards tried to reinterpret this set of equations in terms of vector analysis, spinors *versus* multi-vectors (Parra, 1996). The novelty of Parra's approach is, that he succeeded in formulating tensorial equations without any complexification and thereby no doubling of degrees of freedom. This is achieved by a simple inspection of the resulting eight real equations. Under the assumption, that the real part $\Re(\Psi_1)$ of Ψ_1—first component of the $\mathbb{C}_x^4$ Dirac spinor—transforms as a *scalar* quantity, the full set of eight equations admits a vectorial character. The result is at first not satisfactory since some terms remain to be only third components of vectors. By *introducing* the *spin vector* $\vec{n} = (0, 0, \hbar)$ $(= -iS)$, one obtains a full $SO(3)$ rotationally invariant set of vector equations. Denoting the two scalar quantities as α, λ and the two vectorial quantities as $\vec{E} = (E_1, E_2, E_3)$, $\vec{B} = (B_1, B_2, B_3)$ one arrives at the Parra type $\{0\}$ spinor

$$\Psi_{\{0\}} = \begin{pmatrix} \alpha + iB_3 \\ -B_2 + iB_1 \\ E_3 + i\lambda \\ E_1 + iE_2 \end{pmatrix} \tag{13}$$

It is purely a matter of choice which type of vector component—scalar, first, second, or third vector component—one asserts for $\Re(\Psi_1)$. The other three possibilities yield by the same procedure, also introducing the spin-vector $\vec{n}$ equally well suited spinor–tensor translations. A suitable *choice* of names for the involved scalars and vectors yields:

$$\Psi_{\{0\}} = \begin{pmatrix} \alpha + iB_3 \\ -B_2 + iB_1 \\ E_3 + i\lambda \\ E_1 + iE_2 \end{pmatrix} \Psi_{\{2\}} = \begin{pmatrix} B_2 + iB_1 \\ \alpha - iB_3 \\ E_1 - iE_2 \\ -E_3 + i\lambda \end{pmatrix}$$

$$\Psi_{\{1\}} = \begin{pmatrix} E_1 - iE_2 \\ -E_3 + i\lambda \\ B_2 + iB_1 \\ \alpha - iB_2 \end{pmatrix} \Psi_{\{3\}} = \begin{pmatrix} E_3 + i\lambda \\ E_1 + iE_2 \\ \alpha + iB_3 \\ -B_2 + iB_1 \end{pmatrix} \tag{14}$$

If we now introduce a basis $\{e_i\}$ with Clifford algebraic relations $e_i e_j + e_j e_j = 2\eta_{ij}$ and the above notations for m and q, one obtains four *different* equations:

$$
\begin{aligned}
\{2\}\ \nabla\Psi_{\{2\}}\, e_{21} + qA\Psi_{\{2\}} + m\Psi_{\{2\}}\, e_0 &= 0\ e_{\uparrow}^{+} \\
\{0\}\ -\nabla\Psi_{\{0\}}\, e_{21} + qA\Psi_{\{0\}} + m\Psi_{\{0\}}\, e_0 &= 0\ e_{\downarrow}^{+} \\
\{3\}\ \nabla\Psi_{\{3\}}\, e_{21} - qA\Psi_{\{3\}} + m\Psi_{\{3\}}\, e_0 &= 0\ e_{\uparrow}^{-} \\
\{1\}\ -\nabla\Psi_{\{1\}}\, e_{21} - qA\Psi_{\{1\}} + m\Psi_{\{1\}}\, e_0 &= 0\ e_{\downarrow}^{-}.
\end{aligned} \tag{15}
$$

In the second column, we give the identification—due to Parra—with "particles" associated with the corresponding equations. $\pm$ indicates electron or positron where $\uparrow\!\downarrow$ indicates spin up or down—this is a choice—one might exchange the meanings. The second of these equations—Parra option {2}—happens to be the Dirac–Hestenes equation (10) if we identify the $\{e_i\}$ and $\{\gamma_\mu\}$ bases, which thereby includes the spin explicitly. The other three equations are new. Even if they are similar in structure one is not able to remove the relative changes in sign if two or more of these equations are considered at the same time. Once more, we see the right action of the spin-bivector e_{21} and of the velocity vector e_0. One should note, that proceeding from Dirac theory to quantum electrodynamics (QED), it became necessary to introduce particle and antiparticle creation and annihilation operators for each spin polarization. While in QED the formalism takes care of the different types of spinors, a simple complex linear combination—as quite common in Dirac matrix theory—intermingles the different Parra options without any chance to re-obtain them as different equations.

The Parra spinors can easily be put within a quaternion basis. Let 1, $i_k := ie_k$ be a quaternion basis, then the spinors of r-option become $\Psi_r = q_r^1 + iq_r^2$ where $^-$means quaternion conjugation. Since Hestenes spinors are elements of $\mathcal{C}\ell_{1,3}^{+} \subset \mathcal{C}\ell_{1,3} \simeq M_2(\mathbb{H})$, this can be extended to matrix spinors

$$
\Psi_{\{r\}} = \begin{pmatrix} q_r^1 & -\bar{q}_r^{2} \\ \bar{q}_r^{2} & q_r^1 \end{pmatrix} \tag{16}
$$

The 2×2 matrix structure is a matrix representation of the complex structure (1, i).

Since the Hestenes equation is formulated within abstract algebra and not within a representation it is trivially representation independent. But a change of bases has to be not only an algebra isomorphism but moreover a Clifford algebra isomorphism. Only elements of the Clifford–Lipschitz group $\Gamma_{1,3}$ induce such transformations. Denoting the group of even such elements as $\Gamma_{1,3}^{+}$, we expect the quotient $D = \Gamma_{1,3}/\Gamma_{1,3}^{+}$ to be exactly the *discrete* group of transformations which connect the Parra options. Such transformations are beside the identity space inversion, charge conjugation, and time reversal.

We would, thus, submit that the spin Clifford bundle defined by Rodrigues *et al.* (Rodrigues *et al.*, 1996) is a slightly too large structure, since it does not properly distinguish the different particle types of Parra. The "spin-particle" Clifford bundle should consist of equivalence classes of idempotents with respect to an *even* geometrical equivalence relation. The commutator relation and thus the Clifford structure can be seen to be invariant under discrete—or more generally odd—transformation of the Clifford–Lipschitz group (Crumeyrolle, 1990).

We have established the equivalence of Parra's equations—and the spin Clifford bundle—to the Hestenes formulation. We concentrate now on the connection of Hestenes' and Daviau's Clifford algebraic formulations. The Daviau Clifford algebra form of Dirac's equation will correspond directly to Parra option {1} in (14) as will be shown below.

2.4. Equivalence of Space Clifford and Hestenes Formulation

We will calculate the action of the outer automorphism within the even algebra. We compare the γ_0 action with the action of $*$ introduced in (5) on the Daviau spinor (4),

$$\begin{aligned}\phi_D^* &= \sigma_2\overline{\phi}_D\sigma_2\\ &= \begin{pmatrix} a - c + i(g - h) & -d - f + i(e + b)\\ f - d + i(b - e) & a + c + i(-h - g)\end{pmatrix}\\ &= a\mathbb{1} - d\sigma_1 - b\sigma_2 - c\sigma_3 + ei\sigma_1 - fi\sigma_2 + gi\sigma_3 - hi. \end{aligned} \tag{17}$$

Let use the injection $\sigma_i \mapsto \sigma_i \otimes \mathbb{1}$, which gives a 4×4 representation of the space Clifford algebra, we are able to introduce a γ_0 in this representation, thereby identifying Σ and σ elements. However, this is no longer an element of the Clifford algebra. By comparing with (4) we have

$$\begin{aligned}\gamma_0\phi_D^*\gamma_0 &= a\mathbb{1} + d\sigma_1 + b\sigma_2 + c\sigma_3 + ei\sigma_1 - fi\sigma_2 + gi\sigma_3 + hi\\ &= \phi_D, \end{aligned} \tag{18}$$

This might be rewritten as

$$\phi_D^* = \gamma_0\phi_D\gamma_0 \tag{19}$$

and used in the rewriting of the Dirac–Hestenes equation (12) which then yields the Pauli or space Clifford algebraic equation

$$\sum_\mu \partial_\mu \Psi_H i \sum_3 = -m\Psi_H^* + q\sum_\mu A_\mu \Psi_H. \tag{20}$$

To obtain the full equivalence between this formulation of the Dirac–Hestenes theory to the space Clifford algebraic version of Daviau, we have to perform two further steps.

The first is to explain the additional minus sign in front of the mass term. Redefining the sign of charge and angular momentum measurement, *i.e.*, $e \mapsto -e$, $\hbar \mapsto -\hbar$, results in the appropriate change. Of course, from a particle point of view this two particles are *not* identical. They have a relation as a spin up electron to a spin down positron and do correspond to different types of Parra options in rewriting Hestenes' theory (Parra, 1992). Since no weak interactions are involved here, one physically cannot distinguish these options and there is no harm in these settings. However, one should note that Daviau got four different equations within his calculations and there may be the chance that one of them fit exactly to Hestenes theory without changing the sign of the mass term.

The second step is a relabeling of base elements in a cyclic way. This can be done by defining

$$
\begin{aligned}
z : \sigma &\mapsto \sum \\
z(\mathbb{1}) &= \mathbb{1} \\
z(\sigma_i) &= \textstyle\sum_{i-1} \text{ cyclic.}
\end{aligned}
\tag{21}
$$

The map z can be extended as an outer-morphism, that is a grade preserving extension (Hestenes *et al.*, 1984), to the whole algebra by setting $z(\sigma_i\sigma_j) = z(\sigma_i)z(\sigma_j)$, *etc.* Since z is a cyclic permutation, we have $z^3 = 1$ and $z^{-1} = z^2$. It is crucial to note, that even if in the definition of the * morphism in (5) via complex conjugation followed by a transformation with σ_2, * is not inner, it commutes with z. That is we have $z(\phi^*) = z(\phi)^*$. The map z is inessential to our argument and added for completenes. Daviau has changed notation in recent work to circumvent this renaming.

We obtain the following isomorphism noticing from (7) and (8) that $z^{-1}(\Psi_H) = \phi_D$ holds:

$$\textstyle\sum_\mu \partial_\mu \Psi_H \, i\sum_3 = -m\Psi_H^* + q\sum_\mu A_\mu \Psi_H \tag{22}$$

acting by $m \mapsto -m$ and z^{-1} results in

$$
\begin{aligned}
&(\sigma_0\partial_0 + \sigma_2\partial_1 + \sigma_3\partial_2 + \sigma_1\partial_3)\,\phi_D i\sigma_1 = \\
&m\phi_D^* + q(\sigma_0 A_0 + \sigma_2 A_1 + \sigma_3 A_2 + \sigma_1 A_3)\phi_D
\end{aligned}
\tag{23}
$$

which results with (5) in

$$\nabla\phi_D i\sigma_1 = m\phi_D^* + qA\phi_D. \tag{24}$$

This proves the equivalence of Daviau's Clifford algebraic and Hestenes' formulation of Dirac's theory.

3. RELATED WORK

There is a notorious revival of the transition between spinor and tensor descriptions of Dirac theory. We mentioned Darwin and Madelung, there are also recent approaches of which we will mention only two more. Based on ideas of Sallhofer (1991), Simulik *et al.* (1998) used a spinor–tensor transition, called there *Maxwell–Dirac isomorphism*. Their formalism is a restriction of the approach developed by Parra, however, not so detailed and pedagogical. A description of geometric electron theory with many citations and critical remarks can be found in Keller (1993).

A further genuine and important approach to the spinor–tensor transition was developed starting probably with Crawford by Lounesto (1997). Lounesto investigated the question of how a spinor field can be reconstructed from known tensor densities. The major characterization is derived, using Fierz–Kofink identities, from elements called *Boomerangs*, because they are able to come back to the spinorial picture. Lounesto's result is a characterization of spinors based on multi-vector relations which unveils a new unknown type of spinor.

The notion of a multi-vector is questionable in Dirac theory (Fauser, 1998) and in general (Fauser, 1999d). The $\mathbb{Z}_n$-grading used to define multi-vectors is *not* a feature of Clifford algebra. One expects very different spinor structures if different $\mathbb{Z}_n$-gradings are properly implemented (Fauser *et al.*, 1999b; Fauser, 1997).

4. CONCLUSION

We discussed the isomorphism between spinor and multi-vector formulation of Dirac theory. We proved the equivalence of Daviau's Clifford algebraic and Hestenes' operator spinor formulations of Dirac theory as their equivalence to different options of Parra's treatment. In usual formulations the spinor representations are made up from left actions, while Daviau's formulation requires the bi-module structure of left *and* right actions. A detailed analysis of this fact will be given elsewhere (Fauser, 2000a). Regarding isospin, which was sometimes introduced as right action, our analysis shows that one should be very careful in doing so.

The Daviau spinor is of the most general form—most general element in the algebra—and utilizes the full Pauli algebra as representation space. This should be compared with the Hestenes even operator spinors and ideal or column spinors which span the representation space but *not* the algebra itself. It is peculiar at this point carefully to distinguish representations and abstract algebra. In this sense, Daviau's formulation is the most compact formulation.

We gave some references which critically discussed the concept of multi-vectors or $\mathbb{Z}_n$-gradings in Clifford algebras. One knows that different $\mathbb{Z}_n$-gradings can produce quite different spinor modules. This fact renders the unquestioned multi-vector structure as a peculiar one. A careful study of the representation theory and their dependence on gradings in such cases is required.

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